Université Abdelmalek Assaâdi
FST Tanger/ Année 10-11/MIPC/M112

Série de TD n°5

Exercice1

1. Calculer $\int \frac{x^4 - 3x^2 + 2}{x^4 + x^2} dx.$

2. En déduire $\int \frac{\cos^3(t) + \cos^5(t)}{\sin^2(t) + \sin^4(t)} dt.$

Exercice2

1. Calculer $\int \frac{7x - 5}{x^3 + x^2 - 6x} dx.$

2. Calculer $\int \frac{x^2 + 6x - 1}{(x-3)^2 (x-1)} dx.$

Exercice3

1. Décomposer en éléments simples sur $\boldsymbol{R}$ les fonctions rationnelles suivantes :

$$\frac{x^5 + 2}{x^3 - 1}, \quad \frac{1}{(x^2+1)^2 (x^2 - 1)}.$$

2. Donner les primitives de ces fonctions rationnelles.

Exercice4

Calculer

$$a) \int \frac{1}{\sqrt{(x-1)(2-x)}} dx, \quad b) \int \frac{1}{\sqrt{-2x^2 - 5x - 2}} dx,$$
$$c) \int \frac{1}{(x^2 + a^2)\sqrt{a^2 - x^2}} dx \quad ; a > 0.$$

Exercice5

Calculer les primitives suivantes et indiquer là où les résultats trouvés sont valables:

$$a) \int \frac{1}{\sqrt{e^x + 1}} dx, \quad b) \int \frac{1}{\sqrt{e^x - 1}} dx,$$
$$c) \int \frac{1}{\sqrt{x(1-x)}} dx.$$

Exercice6

1. Calculer $\int_0^{\frac{\pi}{4}} \frac{1}{a + b\cos^2(x)} dx;\ a, b > 0.$

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2. Calculer

$$a)\int_0^x \frac{1}{\sin^2(t)+3\cos^2(t)}dt,\quad b)\int_0^x \frac{1}{2+\cos(t)}dt,$$
$$c)\int_0^x \frac{t^2}{(t\sin(t)+\cos(t))^2}dt.$$

Exercice7

Etudier la convergence des intégrales généralisées suivantes:

$$a)\int_0^{+\infty} \frac{1}{t^\alpha(1+t)^\beta}dt,\quad b)\int_0^{+\infty} \frac{\sin t}{t}dt,$$
$$c)\int_1^{+\infty} \frac{e^{\sin t}}{t}dt,\qquad d)\int_1^{+\infty} \frac{(-1)^{[x]}}{t}dt.$$

Exercice8

1. Etudier la convergence des intégrales suivantes:

$$a)\int_a^b \frac{dt}{\sqrt{(t-a)(b-t)}},\quad b)\int_0^{+\infty} \frac{dt}{(1+t^2)^n},$$
$$c)\int_0^{\frac{\pi}{2}} \ln(\sin t)\,dt.$$

2. Calculer ces intégrales.

Exercice9

On considère les suites $(u_n)_n$ et $(v_n)_n$ définies par:

$$u_n=\sum_{k=1}^{n}\frac{k^\alpha}{n^\alpha+1};\ \alpha\geq 0,\quad v_n=\frac{1}{n}\prod_{k=1}^{n}(n+k)^{\frac{1}{n}}.$$

Calculer leurs limites en utilisant convenablement les sommes de Riemann.

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Exercice 1

• $f(x) = \dfrac{x^4-3x^2+2}{x^4+x^2} = \dfrac{x^4+x^2-4x^2+2}{x^4+x^2} = 1 + \dfrac{-4x^2+2}{x^4+x^2} = 1 + \dfrac{-4x^2+2}{x^2(x^2+1)} = 1 + \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{cx+d}{x^2+1}$

$= 1 + \dfrac{ax(x^2+1)+b(x^2+1)+x^2(cx+d)}{x^2(x^2+1)} = 1 + \dfrac{(a+c)x^3+(b+d)x^2+ax+b}{x^2(x^2+1)}$

$\Rightarrow \begin{cases} a+c=0 \\ b+d=-4 \\ a=0 \\ b=2 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=2 \\ c=0 \\ d=-6 \end{cases}$ d'où $f(x) = 1 + \dfrac{2}{x^2} - \dfrac{6}{x^2+1}$

et $\int f(x)\,dx = x - \dfrac{2}{x} - 6\,\text{Arctan}\,x + C$

• On a : $\dfrac{\cos^3 t+\cos^5 t}{\sin^2 t+\sin^4 t} = \dfrac{\cos t\cdot\cos^2 t\cdot(1+\cos^2 t)}{\sin^2 t+\sin^4 t} = \dfrac{(1-\sin^2 t)(2-\sin^2 t)}{\sin^2 t+\sin^4 t}\cdot\cos t$

On pose $x = \sin t$ d'où $dx = \cos t\cdot dt$

$\int \dfrac{\cos^3 t+\cos^5 t}{\sin^2 t+\sin^4 t}\,dx = \int \dfrac{(1-\sin^2 t)(2-\sin^2 t)}{\sin^2 t-\sin^4 t}\cos t\,dt = \int \dfrac{(1-x)(2-x^2)}{x^2+x^4}\cdot dx = \int f(x)\,dx = x - \dfrac{2}{x} - 6\,\text{Arctan}\,x + C$

alors

$= \sin t - \dfrac{2}{\sin t} - 6\,\text{Arctan}(\sin t) + C$

Exercice 2

a/ $f(x) = \dfrac{7x-5}{x^3+x^2-6x} = \dfrac{7x-5}{x(x^2+x-6)} = \dfrac{7x-5}{x(x-2)(x+3)} = \dfrac{a}{x} + \dfrac{b}{x-2} + \dfrac{c}{x+3} = \dfrac{a(x-2)(x+3)+bx(x+3)+cx(x-2)}{x(x-2)(x+3)}$

Par identification on trouve $a = 5/6$; $b = 9/10$ et $c = -\dfrac{26}{15}$

$f(x) = \dfrac{5/6}{x} + \dfrac{9/10}{x-2} + \dfrac{-\frac{26}{15}}{x+3}$ d'où $\int f(x)\,dx = \dfrac{5}{6}\ln|x| + \dfrac{9}{10}\ln|x-2| - \dfrac{26}{15}\ln|x+3| + C$

b/ $g(x) = \dfrac{x^2+6x-1}{(x-3)^2(x-1)} = \dfrac{a}{x-3} + \dfrac{b}{(x-3)^2} + \dfrac{c}{x-1} = \dfrac{a(x-3)(x-1)+b(x-1)+c(x-3)^2}{(x-3)^2(x-1)} \Rightarrow \begin{cases} a=-1/2 \\ b=13 \\ c=3/2 \end{cases}$

$\int g(x)\,dx = \int -\dfrac{1}{2}\cdot\dfrac{1}{x-3} + 13\cdot(x-3)^{-2} + \dfrac{3}{2}\cdot\dfrac{1}{x-1}\,dx = -\dfrac{1}{2}\ln|x-3| - \dfrac{13}{x-3} + \dfrac{3}{2}\ln|x-1| + C$

Exercice 3

a/ $f(x) = \dfrac{x^5+2}{x^3-1} = \dfrac{x^5-x^2+x^2+2}{x^3-1} = \dfrac{x^5-x^2}{x^3-1} + \dfrac{x^2+2}{x^3-1} = x^2 + \dfrac{x^2+2}{(x-1)(x^2+x+1)} = x^2 + \dfrac{a}{x-1} + \dfrac{bx+c}{x^2+x+1}$

$= x^2 + \dfrac{a(x^2+x+1)+(x-1)(bx+c)}{(x^3-1)} = x^2 + \dfrac{(a+b)x^2+(a-b+c)x+a-c}{x^3-1} \Rightarrow \begin{cases} a+b=1 \\ a-b+c=0 \\ a-c=2 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=0 \\ c=-1 \end{cases}$

$\int f(x)\,dx = \int x^2 + \dfrac{1}{x-1} - \dfrac{1}{x^2+x+1}\,dx = \int x^2 + \dfrac{1}{x-1} - \dfrac{|x+1/2|'}{(x+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx = \dfrac{x^3}{3} + \ln|x-1| - \dfrac{2\sqrt{3}}{3}\,\text{Arctan}\left(\dfrac{2x+1}{\sqrt{3}}\right) + C$

b/ $g(x) = \dfrac{1}{(x^2+1)(x^2-1)} = \dfrac{1}{(x^2+1)(x-1)(x+1)} = \dfrac{ax+b}{x^2+1} + \dfrac{c}{x-1} + \dfrac{d}{x+1} \Rightarrow \begin{cases} a=0 \\ b=-1/2 \\ c=1/4 \\ d=-1/4 \end{cases}$

$\int g(x)\,dx = \dfrac{-1/2}{x^2+1} + \dfrac{1/4}{x-1} + \dfrac{-1/4}{x+1} = -\dfrac{1}{2}\,\text{Arctan}\,x + \dfrac{1}{4}\ln|x-1| - \dfrac{1}{4}\ln|x+1| + C$

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Exercice 4

a/ On a : $(x-1)(2-x) = -x^2+3x-2 = -(x^2-3x+2) = -\left(\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+2\right) = \frac{1}{4}-\left(x-\frac{3}{2}\right)^2$

On pose $x-\frac{3}{2} = \frac{1}{2}u$ alors $dx = \frac{1}{2}du$ et :

$$\int \frac{1}{\sqrt{(x-1)(2-x)}}\,dx = \int \frac{\frac{1}{2}du}{\sqrt{\frac{1}{4}-\frac{1}{4}u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \operatorname{Arcsin} u + C = \operatorname{Arcsin}(2x-3)+C$$

b/ On a : $-2x^2-5x-2 = -2\left(x^2+\frac{5}{2}x+1\right) = -2\left(\left(x+\frac{5}{4}\right)^2-\frac{25}{16}+1\right) = 2\left(\frac{9}{16}-\left(x+\frac{5}{4}\right)^2\right)$

On pose $x+\frac{5}{4} = \frac{3}{4}u$; $dx = \frac{3}{4}du$

$$\int \frac{dx}{\sqrt{-2x^2-5x-2}} = \int \frac{\frac{3}{4}du}{\sqrt{2\left(\frac{9}{16}-\frac{9}{16}u^2\right)}} = \int \frac{\frac{3}{4}du}{\frac{3}{4}\sqrt{2}\sqrt{1-u^2}} = \frac{\sqrt{2}}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{\sqrt{2}}{2}\operatorname{Arcsin} u + C$$
$$= \frac{\sqrt{2}}{2}\operatorname{Arcsin}(4x+5)+C$$

c/ On pose $x = a\sin t$; $dx = a\cos t\,dt$

$$I = \int \frac{dx}{(x^2+a^2)\sqrt{a^2-x^2}} = \int \frac{a\cos t\,dt}{(a^2\sin^2 t+a^2)\sqrt{a^2-a^2\sin^2 t}} = \frac{1}{a^2}\int \frac{1}{1+\sin^2 t}\,dt = \frac{1}{a^2}\int \frac{1}{1+\frac{1+\cos 2t}{2}}\,dt$$
$$= \frac{2}{a^2}\int \frac{1}{3+\cos 2t}\,dt$$

On pose $u = \tan t$ alors $du = (1+u^2)\,dt$

et $\cos 2t = \frac{1-u^2}{1+u^2}$

$$I = \frac{2}{a^2}\int \frac{1}{3+\frac{1-u^2}{1+u^2}}\,\frac{du}{1+u^2} = \frac{2}{a^2}\int \frac{1}{4+2u^2}\,du = \frac{1}{a^2}\int \frac{1}{u^2+(\sqrt{2})^2}\,du = \frac{1}{\sqrt{2}.a^2}\operatorname{Arctan}\frac{u}{\sqrt{2}}+C$$
$$= \frac{1}{a^2\sqrt{2}}\operatorname{Arctan}\left(\frac{\tan t}{\sqrt{2}}\right)+C = \frac{1}{a^2\sqrt{2}}\operatorname{Arctan}\left(\frac{\tan\left(\operatorname{Arcsin}\frac{x}{a}\right)}{\sqrt{2}}\right)$$

Exercice 5

a/ On pose $u = \sqrt{e^x+1} \Rightarrow u^2 = e^x+1$; $2u\,du = e^x\,dx = (u^2-1)\,dx$; $dx = \frac{2u\,du}{u^2-1}$

$$\int \frac{dx}{\sqrt{e^x+1}} = \int \frac{\frac{2u\,du}{u^2-1}}{u} = 2\int \frac{2\,du}{u^2-1} = \int \frac{1}{u-1}-\frac{1}{u+1}\,du = \ln|u-1|-\ln|u+1|+C$$
$$= \ln\left|\frac{u-1}{u+1}\right|+C = \ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right|+C$$

b/ On pose $u = \sqrt{e^x-1}$ $u^2 = e^x-1$ $2u\,du = e^x\,dx \Rightarrow dx = \frac{2u\,du}{1+u^2}$

$$\int \frac{dx}{\sqrt{e^x-1}} = \int \frac{\frac{2u\,du}{1+u^2}}{u} = \int \frac{2}{1+u^2}\,du = 2\operatorname{Arctan} u + C = 2\operatorname{Arctan}(\sqrt{e^x-1})+C$$

c/ $x(1-x) = x-x^2 = -(x^2-x) = -\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right) = \frac{1}{4}-\left(x-\frac{1}{2}\right)^2$

On pose $\left(x-\frac{1}{2}\right) = \frac{1}{2}u$ $dx = \frac{1}{2}du$

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{\frac{1}{2}du}{\sqrt{\frac{1}{4}-\frac{1}{4}u^2}} = \int \frac{du}{\sqrt{1-u^2}} = \operatorname{Arcsin} u + C = \operatorname{Arcsin}(2x-1)+C$$

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Exercice 6

1/ $I = \int_0^{\pi/4} \frac{1}{a+b\cos^2 x}\,dx \quad , a,b>0$

$I = \int_0^{\pi/4} \frac{1}{a+b.\frac{1+\cos 2x}{2}}\,dx = \int_0^{\pi/4} \frac{2}{2a+b+b\cos 2x}\,dx$, On pose $u = \tan x$, $du = (1+u^2)dx$

$I = \int_0^{1} \frac{2}{2a+b+b.\frac{1-u^2}{1+u^2}} \frac{du}{1+u^2} = \int_0^{1} \frac{2}{2a+b+2au^2+bu^2+b-bu^2}\,du = \int_0^{1} \frac{2}{2a+2b+2au^2}\,du$

$= \int_0^{1} \frac{1}{a+b+au^2}\,du = \frac{1}{a}\int_0^{1} \frac{1}{u^2+\frac{a+b}{a}}\,du = \frac{1}{a}.\sqrt{\frac{a}{a+b}}\left[\text{Arctan}\frac{u}{\sqrt{\frac{a+b}{a}}}\right]_0^1 = \frac{1}{\sqrt{a^2+ab}}\,\text{Arctan}\sqrt{\frac{a}{a+b}}$

2/ a/ $I = \int_0^{x} \frac{1}{\sin^2 t+3\cos^2 t}\,dt = \int_0^{x} \frac{1}{\frac{1-\cos 2t}{2}+3.\frac{1+\cos 2t}{2}}\,dt = \int_0^{x} \frac{1}{2+\cos 2t}\,dt$

On pose $u = \tan t$, $du = (1+u^2)\,dt$

$I = \int_0^{\tan x} \frac{1}{2+\frac{1-u^2}{1+u^2}}.\frac{du}{1+u^2} = \int_0^{\tan x} \frac{1}{3+u^2}\,du = \frac{1}{\sqrt{3}}\left[\text{Arctan}\frac{u}{\sqrt{3}}\right]_0^{\tan x} = \frac{\sqrt{3}}{3}\,\text{Arctan}\left(\frac{\tan x}{\sqrt{3}}\right)$

b/ $J = \int_0^{x} \frac{1}{2+\cos t}\,dt$; On pose $u = \tan\frac{t}{2}$ $du = \frac{1}{2}(1+u^2)\,dt$

$J = \int_0^{\tan\frac{x}{2}} \frac{1}{2+\frac{1-u^2}{1+u^2}}.\frac{2du}{1+u^2} = \int_0^{\tan\frac{x}{2}} \frac{2}{3+u^2}\,du = \frac{2}{\sqrt{3}}\left[\text{Arctan}\frac{u}{\sqrt{3}}\right]_0^{\tan\frac{x}{2}} = \frac{2\sqrt{3}}{3}\,\text{Arctan}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)$

Intégrale généralisée

1/ Définition : Une integrale est généralisée si elle est de la forme :

$I = \int_a^{+\infty} f(x)\,dx$ (f definie sur $[a,+\infty[$) ou $I = \int_a^b f(x)\,dx$ (f non definie en a)

2/ Deux integrales de Riemann

a/ $\int_0^1 \frac{1}{x^\alpha}\,dx$ et $\int_a^b \frac{1}{(x-a)^\alpha}\,dx$ convergent $\Leftrightarrow \alpha<1$

b/ $\int_1^{+\infty} \frac{1}{x^\alpha}\,dx$ converge $\Leftrightarrow \alpha>1$

3/ Criteres de convergence

a/ Equivalence : Si $f \sim g$ alors $\int f\,dx$ et $\int g\,dx$ sont de même nature (convergent ou divergent toutes les deux)

b/ Comparaison :

Si $0<f<g$ alors : Si $\int f\,dx$ diverge alors $\int g\,dx$ diverge

Si $\int g\,dx$ converge alors $\int f\,dx$ converge

4/ Calcul direct :

$\int_a^{+\infty} f(x)\,dx = \lim_{A\to+\infty} \int_a^A f(x)\,dx$; $\int_a^b f(x)\,dx = \lim_{t\to a} \int_t^b f(x)\,dx$

(f non definie en a)

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Exercice 7 a/ $I=\int_0^{+\infty}\frac{1}{t^\alpha(1+t)^\beta}\,dt=\int_0^1\frac{1}{t^\alpha(1+t)^\beta}\,dt+\int_1^{+\infty}\frac{1}{t^\alpha(1+t)^\beta}\,dt$

• Au voisinage de 0; $V(0)$: $1+t\sim 1$ et $\frac{1}{t^\alpha(1+t)^\beta}\sim\frac{1}{t^\alpha}$

$\int_0^1\frac{1}{t^\alpha}\,dt$ c.v $\Leftrightarrow \alpha<1$ d'où $\int_0^1\frac{1}{t^\alpha(1+t)^\beta}\,dt$ c.v si $\alpha<1$

. Au voisinage de $+\infty$; $V(+\infty)$: $1+t\sim t$ et $\frac{1}{t^\alpha(1+t)^\beta}=\frac{1}{t^{\alpha+\beta}}$

$\int_1^{+\infty}\frac{1}{t^{\alpha+\beta}}\,dt$ c.v $\Leftrightarrow \alpha+\beta>1$ d'où $\int_1^{+\infty}\frac{1}{t^\alpha(1+t)^\beta}\,dt$ c.v si $\alpha+\beta>1$

Ainsi I converge si $\alpha<1$ et $\alpha+\beta>1$ càd $1-\beta<\alpha<1$

b/ $\int_0^{+\infty}\frac{\sin t}{t}\,dt=\int_0^1\frac{\sin t}{t}\,dt+\int_1^{+\infty}\frac{\sin t}{t}\,dt$

. Au voisinage de 0 ; $V(0)$: $\frac{\sin t}{t}\sim 1$ et $\int_0^1 t\,dt=1$ donc $\int_0^1\frac{\sin t}{t}\,dt$ converge

. $\int_1^A\frac{\sin t}{t}\,dt=\left[-\frac{\cos t}{t}\right]_1^A-\int_1^A\frac{\cos t}{t^2}\,dt$ $\quad\begin{cases}u=\frac{1}{t}\\ v'=\sin t\end{cases}\Rightarrow\begin{cases}u'=-\frac{1}{t^2}\\ v=-\cos t\end{cases}$

$=\cos 1-\frac{\cos A}{A}-\int_1^A\frac{\cos t}{t^2}\,dt$

d'où $\left|\int_1^A\frac{\sin t}{t}\,dt\right|=\left|\cos 1-\frac{\cos A}{A}-\int_1^A\frac{\cos t}{t^2}\,dt\right|\le\cos 1+\frac{1}{A}+\int_1^A\frac{1}{t^2}\,dt$

Par passage à la limite $(A\to+\infty)$ on obtient : $\left|\int_1^{+\infty}\frac{\sin t}{t}\,dt\right|\le\cos 1+\int_1^{+\infty}\frac{1}{t^2}\,dt$

Comme $\int_1^{+\infty}\frac{1}{t^2}\,dt$ converge alors $\int_1^{+\infty}\frac{\sin t}{t}\,dt$ converge d'où $\int_0^1\frac{\sin t}{t}\,dt$ converge

c/ $I=\int_1^{+\infty}\frac{e^{\sin t}}{t}\,dt$: On a $\forall t\in[1,+\infty[$: $-1\le\sin t\le 1$ d'où

$e^{-1}\le e^{\sin t}\le e\Rightarrow\frac{1}{e.t}\le\frac{e^{\sin t}}{t}\Rightarrow\int_1^{+\infty}\frac{1}{e.t}\,dt\le\int_1^{+\infty}\frac{e^{\sin t}}{t}\,dt$

Comme $\int_1^{+\infty}\frac{1}{t}\,dt$ diverge alors I diverge

Exercice 8 1/ $I=\int_a^b\frac{dt}{\sqrt{(t-a)(b-t)}}=\int_a^{\frac{a+b}{2}}\frac{dt}{\sqrt{(t-a)(b-t)}}+\int_{\frac{a+b}{2}}^b\frac{1}{\sqrt{(t-a)(b-t)}}=I_1+I_2$

. Au $V(a)$: $(t-a)(b-t)\sim(t-a)(b-a)$ d'où $\frac{1}{\sqrt{(t-a)(b-t)}}\sim\frac{1}{(t-a)^{1/2}(b-a)^{1/2}}$

$\alpha=\frac{1}{2}<1\Rightarrow\int_a^{\frac{a+b}{2}}\frac{dt}{(t-a)^{1/2}}$ c.v $\Rightarrow I_1$ c.v

. Au $V(b)$: $(t-a)(b-t)\sim(b-a)(b-t)$ d'où $\frac{1}{\sqrt{(t-a)(b-t)}}\sim\frac{1}{(b-a)^{1/2}(b-t)^{1/2}}$

$\alpha=1/2<1\Rightarrow\int_{\frac{a+b}{2}}^b\frac{dt}{(b-t)^{1/2}}$ c.v $\Rightarrow I_2$ c.v

Ainsi I converge

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b/ $J=\int_0^{+\infty}\frac{dt}{(1+t^2)^n}=\int_0^1\frac{dt}{(1+t^2)^n}+\int_1^{+\infty}\frac{dt}{(1+t^2)^n}=I_1+I_2$

- $f(t)=\frac{1}{(1+t^2)^n}$ est continue sur $[0,1]$ donc I_1 converge
- Au $V(+\infty)$: $1+t^2\sim t^2$ donc $\frac{1}{(1+t^2)^n}\sim\frac{1}{t^{2n}}$

$n\geq 1 \Rightarrow 2n\geq 2>1$ donc $\int_1^{+\infty}\frac{1}{t^{2n}}dt$ converge $\Rightarrow I_2$ converge $\Rightarrow I$ converge

c/ $I=\int_0^{\pi/2}\ln(\sin t)\,dt$. Au voisinage de 0 : montrons que $\ln(\sin t)\sim\ln t$

. $\frac{\ln(\sin t)}{\ln t}=\frac{\ln(t\cdot\frac{\sin t}{t})}{\ln t}=\frac{\ln t+\ln\frac{\sin t}{t}}{\ln t}=1+\frac{\ln\frac{\sin t}{t}}{\ln t}\xrightarrow[t\to 0^+]{}1+\frac{0}{-\infty}=1$

Or : $\int_0^{\pi/2}\ln t\,dt=[t\ln t]_0^{\pi/2}-\int_0^{\pi/2}1\,dt=\frac{\pi}{2}\ln\frac{\pi}{2}-[t]_0^{\pi/2}=\frac{\pi}{2}\ln\frac{\pi}{2}-\frac{\pi}{2}<+\infty$ $\begin{cases}u=\ln t\\ v'=1\end{cases}$ $\begin{cases}u'=\frac{1}{t}\\ v=t\end{cases}$

donc I converge

2/ Calcul de $I=\int_a^b\frac{dt}{\sqrt{(t-a)(b-t)}}$; $a<b$

1° méthode : $(t-a)(b-t)=bt-t^2-ab+at=-(t^2-(a+b)t)-ab=-\left[\left(t-\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2\right]-ab$

$=-\left(\left(t-\frac{a+b}{2}\right)^2+\frac{(a+b)^2-4ab}{4}\right)=\frac{(a-b)^2}{4}-\left(t-\frac{a+b}{2}\right)^2$

On pose : $t-\frac{a+b}{2}=\frac{a-b}{2}u$ (On peux poser aussi : $t-\frac{a+b}{2}=\frac{a-b}{2}\cos u$)

alors $dt=\frac{a-b}{2}du$; Si $t=a$ alors $a-\frac{a+b}{2}=\frac{a-b}{2}u\Rightarrow u=1$

Si $t=b$ alors $b-\frac{a+b}{2}=\frac{a-b}{2}u\Rightarrow u=-1$

$I=\int_1^{-1}\frac{\frac{a-b}{2}du}{\sqrt{\frac{(a-b)^2}{4}-\frac{(a-b)^2}{4}u^2}}=\int_1^{-1}\frac{\frac{a-b}{2}du}{\left|\frac{a-b}{2}\right|\sqrt{1-u^2}}=\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}du=[\text{Arc}\sin u]_{-1}^{1}=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$

2° méthode : Examen 05-06 (on propose de poser : $t=a\cos^2x+b\sin^2x$)

Si $t=a$ alors $a=a\cos^2x+b\sin^2x\Rightarrow\cos x=1$ et $\sin x=0\Rightarrow x=0$

Si $t=b$ alors $b=a\cos^2x+b\sin^2x\Rightarrow\cos x=0$ et $\sin x=1\Rightarrow x=\pi/2$

$dt=(-2a\cos x\sin x+2b\sin x\cos x)\,dx=2(b-a)\sin x\cos x\,dx$

$t-a=a\cos^2x+b\sin^2x-a=a(\cos^2x-1)+b\sin^2x=-a\sin^2x+b\sin^2x=(b-a)\sin^2x$

$b-t=b-a\cos^2x-b\sin^2x=b(1-\sin^2x)-a\cos^2x=b\cos^2x-a\cos^2x=(b-a)\cos^2x$

$I=\int_0^{\pi/2}\frac{2(b-a)\sin x\cos x\,dx}{\sqrt{(b-a)\sin^2x\cdot(b-a)\cos^2x}}=\int_0^{\pi/2}2\,dx=[2x]_0^{\pi/2}=\pi$

b/ $I_n=\int_0^{+\infty}\frac{dt}{(1+t^2)^n}$. On pose $\begin{cases}u=\frac{1}{(1+t^2)^n}=(1+t^2)^{-n}\\ v'=1\end{cases}\Rightarrow\begin{cases}u'=-2nt(1+t^2)^{-n-1}=\frac{-2nt}{(1+t^2)^{n+1}}\\ v=t\end{cases}$

$I_n=\left[\frac{t}{(1+t^2)^n}\right]_0^{+\infty}-\int_0^{+\infty}\frac{-2nt^2}{(1+t^2)^{n+1}}dt=2n\cdot\int_0^{+\infty}\frac{t^2}{(1+t^2)^{n+1}}dt=2n\cdot\int_0^{+\infty}\frac{1+t^2}{(1+t^2)^{n+1}}-\frac{1}{(1+t^2)^{n+1}}dt$

$I_n = 2n\left(\int_0 \frac{1}{(1+t^2)^n}\,dt - \int_0 \frac{1}{(1+t^2)^{n+1}}\,dt\right)$; $I_n = 2n(I_n - I_{n+1})$

$I_n = 2n I_n - 2n.I_{n+1} \Rightarrow I_{n+1} = \frac{2n-1}{2n}.I_n$; $\forall n \geq 1$

De plus : $I_1 = \int_0^{+\infty} \frac{1}{1+t^2}\,dt = [\text{Arctan } t]_0^{+\infty} = \pi/2 - 0 = \pi/2$

Remarque : $I_0 = \int_0^{+\infty} 1\,dt = [t]_0^{+\infty} = +\infty$ (n'existe pas)

On a : $\forall n \geq 1$: $I_{n+1} = \frac{2n-1}{2n}.I_n$

- $I_n = \frac{2n-3}{2n-2} I_{n-1}$
- $I_{n-1} = \frac{2n-5}{2n-4} I_{n-2}$
- $\vdots$
- $I_3 = \frac{3}{4} I_2$
- $I_2 = \frac{1}{2} I_1$

On fait leur produit et on obtient :

$I_n = \frac{1\times 3\times \dots\times(2n-3)}{2\times 4\times\dots(2n-2)} I_1$

$I_n = \frac{1\times 2\times 3\times 4\times\dots\times(2n-3)(2n-2)}{(2\times 4\times\dots\times(2n-2))^2}\cdot\frac{\pi}{2}$

$I_n = \frac{(2n-2)!}{(2^{n-1}.(n-1)!)^2}\cdot\frac{\pi}{2} = \frac{\pi(2n-2)!}{2^{2n-1}((n-1)!)^2}$

c/ $I = \int_0^{\pi/2} \ln(\sin t)\,dt$. Considérons 3 intégrales suivantes :

$J = \int_0^{\pi/2} \ln(\cos t)\,dt$; $K = \int_0^{\pi} \ln(\sin t)\,dt$, $L = \int_0^{\pi/2} \ln\sin(2t)\,dt$

- $J = \int_0^{\pi/2} \ln(\cos t)\,dt = \int_{\pi/2}^{0} \ln(\cos(\frac{\pi}{2}-u))(-du) = \int_0^{\pi/2} \ln\sin u\,du = I$; $t = \frac{\pi}{2}-u$
- $K = \int_0^{\pi} \ln(\sin t)\,dt = \int_0^{\pi/2} \ln(\sin t)\,dt + \int_{\pi/2}^{\pi} \ln(\sin t)\,dt = \int_0^{\pi/2} \ln(\sin t)\,dt + \int_{\pi/2}^{0} \ln(\sin(\pi-u))(-du)$; $t =$

$= \int_0^{\pi/2} \ln(\sin t)\,dt + \int_0^{\pi/2} \ln(\sin u)\,du = I + I = 2I$

- $L = \int_0^{\pi/2} \ln(\sin 2t)\,dt = \int_0^{\pi} \ln(\sin u)\cdot\frac{du}{2} = \frac{1}{2}\int_0^{\pi} \ln(\sin u)\,du = \frac{1}{2}(2I) = I$; $u = 2t$

. De plus $L = \int_0^{\pi/2} \ln(\sin 2t)\,dt = \int_0^{\pi/2} \ln(2\sin t.\cos t)\,dt = \int_0^{\pi/2} \ln 2 + \ln\sin t + \ln\cos t\,dt = \int_0^{\pi/2} \ln 2\,dt +$

d'où $I = [t\ln 2]_0^{\pi/2} + I + I \Rightarrow I = -\frac{\pi}{2}\ln 2$

Exercice 9

1/ $U_n = \sum_{k=1}^{n} \frac{k^\alpha}{n^{\alpha+1}} = \frac{1}{n}\sum_{k=1}^{n} \frac{k^\alpha}{n^\alpha} = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^\alpha \longrightarrow \int_0^1 x^\alpha\,dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1}$

2/ $V_n = \frac{1}{n}\prod_{k=1}^{n}(n+k)^{\frac{1}{n}}$; $\ln V_n = \ln\left(\frac{1}{n}\prod_{k=1}^{n}(n+k)^{1/n}\right) = \ln\frac{1}{n} + \ln\prod_{k=1}^{n}(n+k)^{1/n}$

$= -\ln n + \sum_{k=1}^{n}\ln(n+k)^{1/n} = -\ln n + \sum_{k=1}^{n}\frac{1}{n}\ln(n+k) = -\ln n + \sum_{k=1}^{n}\frac{1}{n}\ln n(1+\frac{k}{n}))$

$= -\ln n + \frac{1}{n}\left(\sum_{k=1}^{n}\ln n + \sum_{k=1}^{n}\ln(1+\frac{k}{n})\right) = -\ln n + n\left(\frac{\ln n}{n}\right) + \frac{1}{n}\sum_{k=1}^{n}\ln(1+\frac{k}{n})$

$= \frac{1}{n}\sum_{k=1}^{n}\ln(1+\frac{k}{n}) \xrightarrow{n\to+\infty} \int_0^1 \ln(1+x)\,dx = [x\ln(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$ $\left\{\begin{matrix} u = \ln(1+x) \\ v' = 1 \end{matrix}\right\}$

$= \ln 2 - \int_0^1 1 - \frac{1}{1+x}\,dx = \ln 2 - [x - \ln(1+x)]_0^1 = 2\ln 2 - 1 \Rightarrow \lim_{\infty} V_n = e^{2\ln 2 - 1} = \frac{4}{e}$ ./.

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